Ordinary Differential Equations (ODEs)

- An **Ordinary Differential Equation** (ODE) involves one or more derivatives with respect to a **single independent variable**, usually time (t). For example, this is Newton's Cooling Law:

$$\frac{dT}{dt} = -k(T - T_a)$$

  where
  - $T(t)$ is the unknown, the temperature of the body at time t.
  - T_a is the constant ambient temperature.
  - $k > 0$ is the cooling coefficient.
- ODEs are widely used in Earth Sciences to model:
  - Radioactive decay.
  - Cooling/heating of materials.
  - Chemical reaction kinetics.
  - Mass transfer in simple systems.

ODEs vs PDEs

- **Partial Differential Equations (PDEs)** involve derivatives with respect to **multiple independent variables**, e.g. time and space.

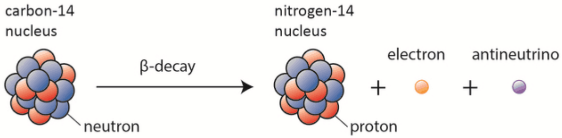
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

- Example: the heat equation in a 1D rock layer ($T = T(x, t)$).
- PDEs are essential to describe:
  - Heat diffusion in the Earth's crust.
  - Groundwater flow.
  - Deformation of rocks under stress.
  - Atmospheric or oceanic dynamics.
  - Conservation laws (mass, momentum, energy).
- Today we will look at some basic numerical method so solve ODEs.

Radioactive Decay in Earth Sciences

Let consider now another example of physical process described by an ODE.

- Radioactive isotopes are used for **radiometric dating** of rocks and minerals.
- Examples:
    - $^{238}\text{U} \rightarrow {}^{206}\text{Pb}$ (half-life $\approx$ 4.5 billion years)
    - $^{14}\text{C} \rightarrow {}^{14}\text{N}$ (half-life $\approx$ 5730 years)



- The amount of the radioactive parent isotope decreases over time.
- We want to model the evolution of the quantity $N(t)$ = number of atoms at time t.

From Discrete Steps to Continuous Model

- Let N_k be the number of atoms at time $t_k = k\Delta t$.
- Assume a constant fraction α of atoms decays in each time step:

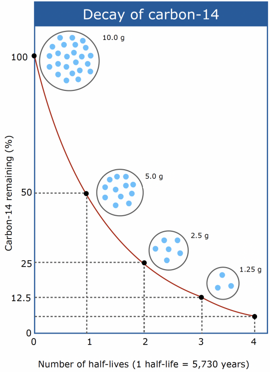
$$N_{k+1} = N_k - \alpha N_k = (1-\alpha)N_k$$

- Recursive expression:

$$N_k = (1-\alpha)^k N_0$$

- Taking the limit as $\Delta t \to 0$ and defining $\lambda = \lim_{\Delta t \to 0} \frac{\alpha}{\Delta t}$:

$$\frac{dN(t)}{dt} = -\lambda N(t)$$



The Differential Equation of Radioactive Decay

- We have derived the ODE:

$$\frac{dN(t)}{dt} = -\lambda N(t)$$

- Notation:
  - $N(t)$: number of atoms (or concentration) at time t.
  - t: time (independent variable).
  - λ: decay constant, specific to the isotope.
- This is a first-order linear ODE with known analytical solution:

$$N(t) = N_0 e^{-\lambda t}$$

- This model is the foundation of many dating techniques in geochronology.

A Simplified Model for Water Infiltration

We model the volumetric water content $\theta(t)$ as it decreases due to percolation. We assume that water is lost from the soil due to gravity-driven flow, with a conductivity depending on the current water content:

$$\frac{d\theta}{dt} = -K(\theta)$$

A commonly used empirical expression for the hydraulic conductivity is:

$$K(\theta) = K_s \cdot \theta^n$$

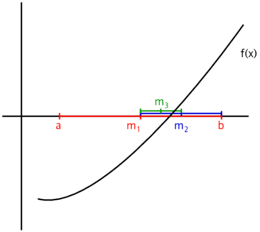
- K_s: saturated conductivity (e.g., 10^{-5} to 10^{-6} m/s)
- $n > 1$: nonlinearity parameter (e.g., $n = 3$ or 4)

Root-Finding: The Bisection Method - Idea

The **bisection method** is a simple and robust numerical technique for finding a root of a continuous function $G(X)$.

Core Idea:

- Start with an interval $[a, b]$ such that $G(a)$ and $G(b)$ have *opposite signs* (i.e., $G(a) \cdot G(b) < 0$).
- By the *Intermediate Value Theorem*, if $G(X)$ is continuous, there must be at least one root X^* in (a, b).
- The method repeatedly halves the interval while keeping the subinterval where the sign change occurs.



Steps visually:

1. Find a, b so $G(a)G(b) < 0$.
2. Calculate midpoint $c = (a + b)/2$.
3. If $G(c) \approx 0$, c is the root.
4. If $G(a)G(c) < 0$, root is in $[a, c]$.
5. Else, root is in $[c, b]$.
6. Repeat with new interval.